Math 2010B Tutorial 6

Outline :

- Partial Derivative

- Differentiability

e.g Partial derivative in Polar wordinate

Let $f(x,y) = xe^{y} + \frac{y}{x}$, $x, y \in \mathbb{R}^2 \setminus \{0\}$

Let (r,θ) be polar cood.

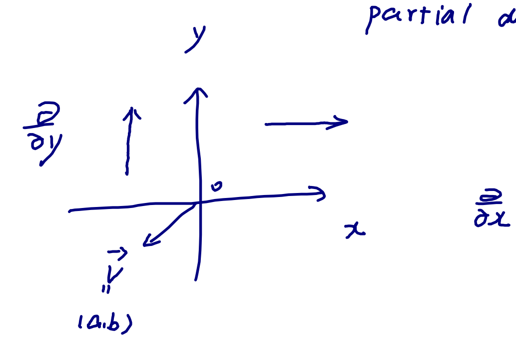
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

1) Find $\frac{\partial f}{\partial r}$ at $(r,\theta) = (1,0)$

2) Find $\frac{\partial f}{\partial \theta}$ at $(r,\theta) = (1,0)$

$$\boxed{\begin{aligned} &\text{remark:} \\ &\frac{\partial f}{\partial \vec{v}} = a \cdot \frac{\partial f}{\partial x} + b\,\frac{\partial f}{\partial y} \end{aligned}}$$

if $\vec{v} = (1,0)$
$\vec{v} = (0,1)$
partial derivative.



Sol: 1) $\frac{\partial f}{\partial r}$.

$$f(r,\theta) = r\cos\theta\, e^{r\sin\theta} + \underline{\tan\theta}$$

$$\Rightarrow \frac{\partial f}{\partial r} \overset{?}{=} \frac{\partial f(r,\theta)}{\partial r} = \frac{\partial}{\partial r}\left(r\cos\theta\, e^{r\sin\theta}\right) + 0 = \cos\theta\, e^{r\sin\theta} + r\sin\theta\cos\theta\, e^{r\sin\theta}$$

$$\underline{\underline{r=1,\ \theta=0}} \quad 1$$

2) $$\frac{\partial f}{\partial \theta} = -r\sin\theta\, e^{r\sin\theta} + r^2\cos^2\theta\, e^{r\sin\theta} + \sec^2\theta$$

$$\underline{\underline{r=1,\ \theta=0}} \quad 2 \qquad (\text{at } (r,\theta) = (1,0))$$

$n=1$

if $f : \mathbb{R} \to \mathbb{R}$ is differentiabe $\Rightarrow$ f is continuous.

$\Leftrightarrow$ $\frac{\partial f}{\partial x}$ exist.

if $f \in C^1 \Leftrightarrow \frac{\partial f}{\partial x_i}$ exist & continuous

$\Downarrow$

$n=2$ $(n \geq 2)$ if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable $\Rightarrow$ f is continuous

$\Downarrow$ $\quad \not\Uparrow$ $\quad$?

$\underline{\frac{\partial f}{\partial x_i} \text{ exist}}$ all $1 \leq i \leq n$ $\not\Rightarrow$

Q: Is there a function $f : \mathbb{R}^n \to \mathbb{R}$ $^{n \geq 2}$ which is non-differentiabe at $\vec{a}$, but all partial derivative of f at $\vec{a}$ exist ?

A: Yes

for $n=2$ Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

for $n=2$ Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Step 1. $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0$

$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0$

i.e $f_x(0,0)$, $f_y(0,0)$ exist ✓

Q whether $f(x,y)$ is differentiable at $(0,0)$?

$\boxed{\text{if } f(x,y) \text{ is differentiable at } (0,0) \Rightarrow f(x,y) \text{ is continuous at } (0,0)}$

$\lim_{(x,y)\to(0,0)} f(x,y) \overset{?}{=}$ $\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{\substack{r\to 0\\ \theta=\frac{\pi}{4}}} \dfrac{r^2 \sin\theta \cos\theta}{r^2} = \sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{2}$

$\neq f(0,0) \Rightarrow f(x,y)$ is not continuous at $(0,0)$

$\Rightarrow f$ is not differentiable at $(0,0)$.

Recall $f(x,y)$ is continous at $\underline{(0,0)}$.

Step 1. $\underline{\lim_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} \overset{?}{\exists}}$

Step 2 i) if $\exists \triangleq c$, $\Rightarrow$ $c \overset{?}{=} f(0,0)$

$\underline{\text{if } c = f(0,0) \quad \text{continuous at } (0,0)}$

if $c \neq f(0,0)$ is continuous at $(0,0)$

ii) if $\nexists$ $\Rightarrow$ $f(x,y)$ is not continuous at $(0,0)$.

Q: Why we set $\theta = \frac{\pi}{4}$ in example?
$x = y$

Answer: if $\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) \; \exists \; \neq \; f(0,0)$

$\Rightarrow$ $f(x,y)$ is not continuous

choose two path (compare)

$\lim_{\substack{(x,y)\to(0,0)\\ \text{path}_1}} f(x,y) \neq \lim_{\substack{(x,y)\to(0,0)\\ \text{path}}} f(x,y)$

$\Rightarrow$ $f(x,y)$ is not continuous e.g. Show that a linear Polynomial

$$f(\vec{x}) = c + b_1 x_1 + \cdots + b_n x_n \quad (\text{here } \vec{x} = (x_1, \ldots, x_n))$$

is differentiable on $\mathbb{R}^n$ from definition.

Sol: Fix $\vec{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$

1° check if all partial derivative exist.

Fix $i \in \{1, \ldots n\}$, $\frac{\partial f}{\partial x_i}(\vec{a}) = b_i$.

2° Examine the error term.

at $\vec{a}$. the error term is

$$\varepsilon(\vec{x}) = f(\vec{x}) - f(\vec{a}) - \underline{\sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{= b_i} (x_i - a_i)}$$

$$= \sum_{i=1}^{n} b_i (x_i - a_i) - \sum_{i=1}^{n} b_i (x_i - a_i) = 0$$

$\therefore$ Obviously: $\lim_{\vec{x} \to \vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0 \Rightarrow f$ is differentiable at $\vec{a}$

As $\vec{a}$ is arbitrary $\Rightarrow f$ is differentible on $\mathbb{R}^n$.

Remark: Step 1 $\frac{\partial f}{\partial x_i} = b_i$ constant

Step 2. $\frac{\partial f}{\partial x_i}$ is continuous for all i

Step 3: f is differentiable on $\mathbb{R}^n$.